More Building on Taking the Derivative of an Integral

Recall FTC Part 1: If $g(x)=\int_a^x f(t)\,dt$, then $g'(x)=f(x)$.

Recall: $\frac{d}{dx}\int_1^x t^2\,dt$ $= x^2 \cdot x'$ ← prime

$= x^2 \cdot 1 = \boxed{x^2}$

Recall: when upper bound is more complex than x: $\frac{d}{dx}\int_1^{\sin x} t^2\,dt = \sin^2 x \cdot (\sin x)'$

$= \boxed{\sin^2 x \cdot \cos x}$

Recall: when constant is the upper bound: $\frac{d}{dx}\int_{\sin x}^1 t^2\,dt$

$= -\frac{d}{dx}\int_1^{\sin x} t^2\,dt$

$= \boxed{-\sin^2 x \cdot \cos x}$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Do: $\frac{d}{dx}\int_{3x}^4 \left(7t^2 - t + 2\right)\,dt$

answer should not contain parentheses

$= -\frac{d}{dx}\int_4^{3x}(7t^2-t+2)\,dt$

$= -\left(7(3x)^2 - 3x + 2\right)\cdot 3$ ← $(3x)'$

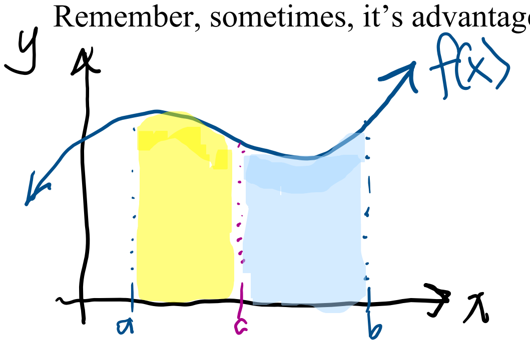
$= -3\left(7\cdot 9x^2 - 3x + 2\right)$

$= -3\left(63x^2 - 3x + 2\right)$

$63 \times 3 = 189$

$= \boxed{-189x^2 + 9x - 6}$

Remember, sometimes, it's advantageous to split the integral into two pieces.



$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Let's apply this to an FTC Pt 1 problem:

split into two integrals:

ex. $\frac{d}{dx}\int_{x^2}^{4x^2} \ln\sqrt{t}\, dt$

Problem: variables in both bounds

$$= \frac{d}{dx}\left(\int_{x^2}^{0} \ln\sqrt{t}\, dt + \int_{0}^{4x^2} \ln\sqrt{t}\, dt\right)$$

$$= \frac{d}{dx}\int_{x^2}^{0} \ln\sqrt{t}\, dt + \frac{d}{dx}\int_{0}^{4x^2} \ln\sqrt{t}\, dt$$

$$= -\frac{d}{dx}\int_{0}^{x^2} \ln\sqrt{t}\, dt + \ln\sqrt{4x^2}\cdot 8x$$

$$= -\ln\sqrt{x^2}\cdot 2x + \ln\sqrt{(2x)^2}\cdot 8x \qquad -a+b=b-a$$

$\sqrt{4x^2}=b \qquad \sqrt{x^2}=a$

$$= \boxed{8x\ln(2x) - 2x\ln x}$$

$F(b) - F(a)$

Let's streamline the previous problem using FTC Pt 2:

Revisit $\frac{d}{dx}\int_{x^2}^{4x^2} \ln\sqrt{t}\, dt = F(b) - F(a)$

$$= \ln\sqrt{4x^2}\cdot 8x - \ln\sqrt{x^2}\cdot 2x$$

$$= \boxed{8x\ln(2x) - 2x\ln x}$$

$F(b)\cdot b' - F(a)\cdot a'$

Do: $\frac{d}{dx}\int_{x^3}^{\sin x} e^t\, dt = e^{\sin x}\cdot\cos x - e^{x^3}\cdot 3x^2$

$$= \boxed{\cos x\cdot e^{\sin x} - 3x^2 e^{x^3}}$$

$e^{2x} = (e^x)^2$ then $\sqrt{(e^x)^2} = e^x$

ex: $\frac{d}{dx}\int_{e^{2x}}^{\sec^2 x}\sqrt{t}\ dt = \sqrt{\sec^2 x}\cdot\left[(\sec x)^2\right]' - \sqrt{e^{2x}}\cdot\left(e^{2x}\right)'$

$= 2\sec x\cdot\sec^2 x\tan x - e^x\cdot 2e^{2x}$

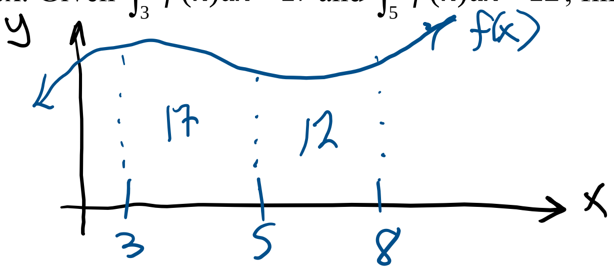
$= \boxed{2\sec^3 x\tan x - 2\underbrace{e^x e^{2x}}_{e^{3x}}}$

$\left[(\sec x)^2\right]'$
$= 2u\cdot u'$ $\quad u = \sec x$, $u' = \sec x\tan x$
$= 2\sec x\cdot\sec x\tan x$
$= 2\sec^2 x\cdot\tan x$

$x^a x^b = x^{a+b}$

Another application using the splitting of integrals:

ex. Given $\int_3^5 f(x)dx = 17$ and $\int_5^8 f(x)dx = 12$, find $\int_3^8 f(x)dx$. $= 17 + 12 = \boxed{29}$



ex. Given $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$, find $\int_8^{10} f(x)dx$.

$$\int_0^{10} f(x)dx = \int_0^8 f(x)dx + \int_8^{10} f(x)dx$$

$$17 = \cancel{12} + \int_8^{10} f(x)dx \quad \Rightarrow \int_8^{10} f(x)dx = 5$$

$$-12 \qquad -\cancel{12}$$

ex. Given $\int_0^{10} f(x)dx = 8$, find $\int_0^{10}(3f(x)+5)dx$. apply integral to each term

$= \int_0^{10} 3\cdot f(x)dx + \int_0^{10} 5\,dx$

$= 3\int_0^{10} f(x)dx + 5x\Big|_0^{10}$

$= 3(8) + 5(10-0)$

$= 24 + 50 = \boxed{74}$

$(\arctan x)' = \frac{1}{1+x^2}$

Last Time: $\int \frac{\arctan x}{1+x^2}dx = \int \arctan x \cdot \frac{1}{1+x^2}dx$

$= \int u \, du$

$= \frac{u^2}{2} + C$

$= \boxed{\frac{1}{2}(\arctan x)^2 + C}$

$u = \arctan x$

$du = \frac{1}{1+x^2}dx$

Compare: $\int \frac{x}{1+x^2}dx = \int \frac{1}{1+x^2} \cdot x\,dx$

$= \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|1+x^2| + C$

$= \boxed{\frac{1}{2}\ln(1+x^2) + C}$

$u = 1+x^2$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$(\ln x)' = \frac{1}{x}$

$x^2 > 0$

$1+x^2 > 0$

More Compare and Contrast:

Do: $\int \sqrt{1+x^2}\; x\,dx$

$= \frac{1}{2}\int u^{1/2}\,du$

$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2} + C$

$= \boxed{\frac{1}{3}(1+x^2)^{3/2} + C}$

$u = 1+x^2$

$\frac{1}{2}du = x\,dx$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$

$\frac{1}{\frac{3}{2}} \overset{KCF}{=} 1 \cdot \frac{2}{3}$

$a(b-c)$

$ab - ac$

However: $\int \sqrt{1+x}\; x\,dx$

$= \int \sqrt{u}\; x\; du$

$= \int u^{1/2}(u-1)\,du$

$= \int (u^{3/2} - u^{1/2})\,du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \Rightarrow \boxed{\frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C}$

$u = 1+x$

$du = 1 \cdot dx \Rightarrow du = dx$

$u = 1+x$ solve for x

$u - 1 = x$